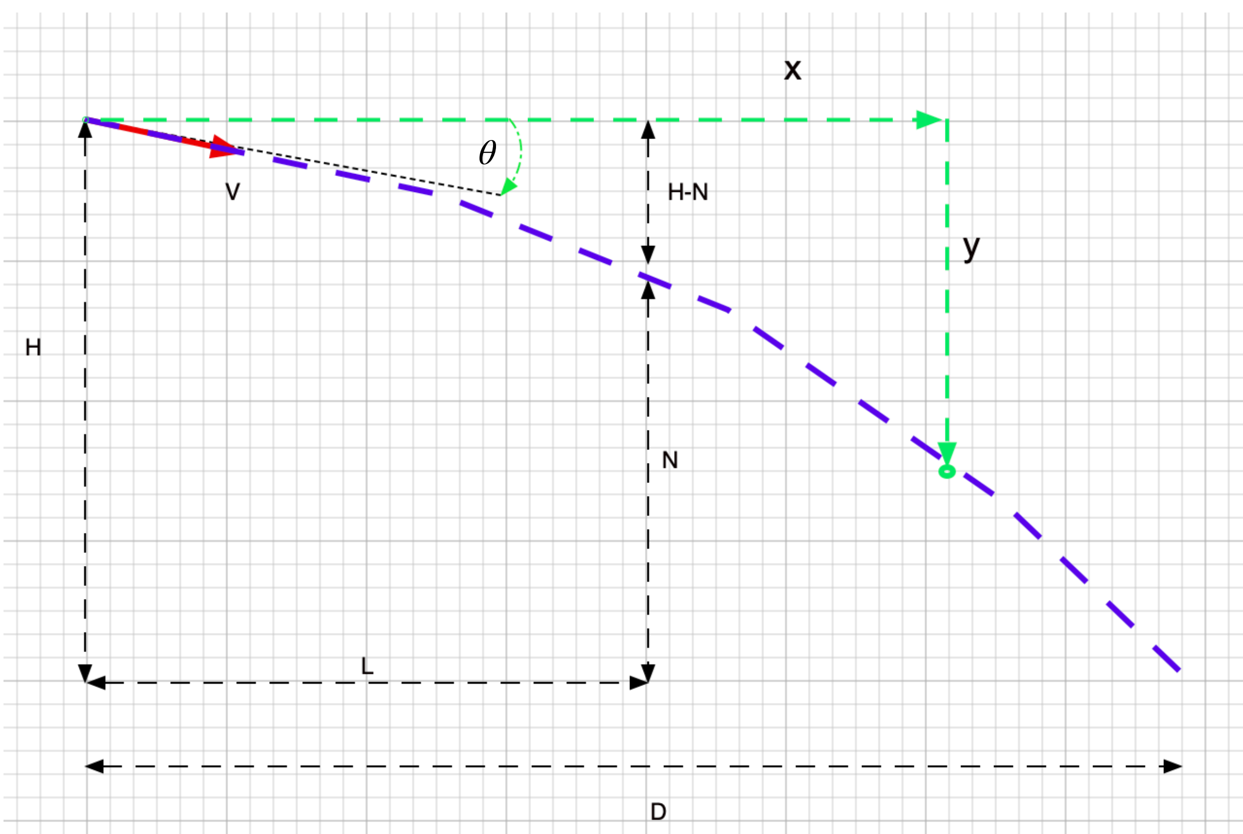
Serving at a speed of 170 km/h, a tennis player hits the ball at a height of 2.5 m and an angle θ below the horizontal. The baseline from which the ball is served is 11.9 m from the net, which is 0.91 m high. What is the angle θ such that the ball just crosses the net? Will the ball land in the service box, which has an outermost service line 6.40 m from the net?

Assumptions:

> Assume that the friction force of the ball moving through the air is negligible. After imparting the initial velocity V to the ball, the only force acting on the ball is the gravitational force.

Variables:

> At time t, the position of the tennis ball (green) is given by its coordinates x(t) and y(t). Note that positive values of x are to the right, while positive values of y are down. Initially the magnitude of the velocity of the tennis ball is at t=0 is given by V. The velocity vector is inclined at an angle θ below the horizon. This angle is positive when measured clockwise as indicated in the diagram.



We need to have consistent units in our problem. Our first step is to convert 170km/hr into m/s

Newton's Second Law states that the force applied to a body is equal to its mass times its acceleration. Forces, velocities and accelerations are vectors having components (in this case) in the x and y directions. The only force acting on the tennis ball is gravity which is in the vertical direction. That means there is no acceleration in the horizontal direction and the velocity of the tennis ball (in the horizontal direction) remains unchanged from its initial value. If the magnitude of that velocity is V, then the horizontal component of the velocity $v_x(t)$ is given by

(1) $$v_x = V\cos(\theta)$$

The time for the ball to strike the plane of the net is the distance to the net (L) divided by the horizontal component of the velocity

(2) $$t_n = \frac{L}{V\cos\theta}$$

In the vertical direction the force of gravity acts on the tennis ball, this causes the arc of the trajectory to curve downward. While the initial velocity (downward) of the ball is $v_y = V\sin(\theta)$ the downward speed will increase due to gravitational acceleration. Newton's law for the acceleration in the vertical direction can be written

(3) $$Mg = M\frac{dv_y}{dt}$$

Here, the mass of the tennis ball cancels on both sides of the equation yielding the result that the change in velocity is independent of the mass. Integrating equation (3) leads to the result

(4) $$v_y(t) = V\sin(\theta) + gt$$

where the first term on the right hand side is the initial vertical velocity. Integrating equation (4) a second time we get the familiar equation for the distance that an object falling in a gravitation field will travel.

(5) $$y(t) = tV\sin(\theta) + \frac{1}{2}gt^2$$

The time at which the ball passes the plane of the net is given by (2). Therefore, the vertical distance of the ball at the time it passes the net is found by using t_n from equation (2) and substituting into (5). If at this time y=H-N (the ball just passes over the net then we get the following equation.

(6) $$H - N = \frac{L}{V\cos(\theta)}V\sin(\theta) + \frac{L^2 g}{2V^2\cos^2(\theta)}$$

Just as a check to see if equation (6) is correct, let's look at the units of each of the three terms. If expressed properly each of the three terms has the units of distance.

Let's rewrite (6) in a slightly different form.

(7) $$A\cos^2(\theta) - B\sin(\theta)\cos(\theta) - C = f(\theta) = 0$$

$$A = H - N = 2.5m - .91m = 1.59m$$

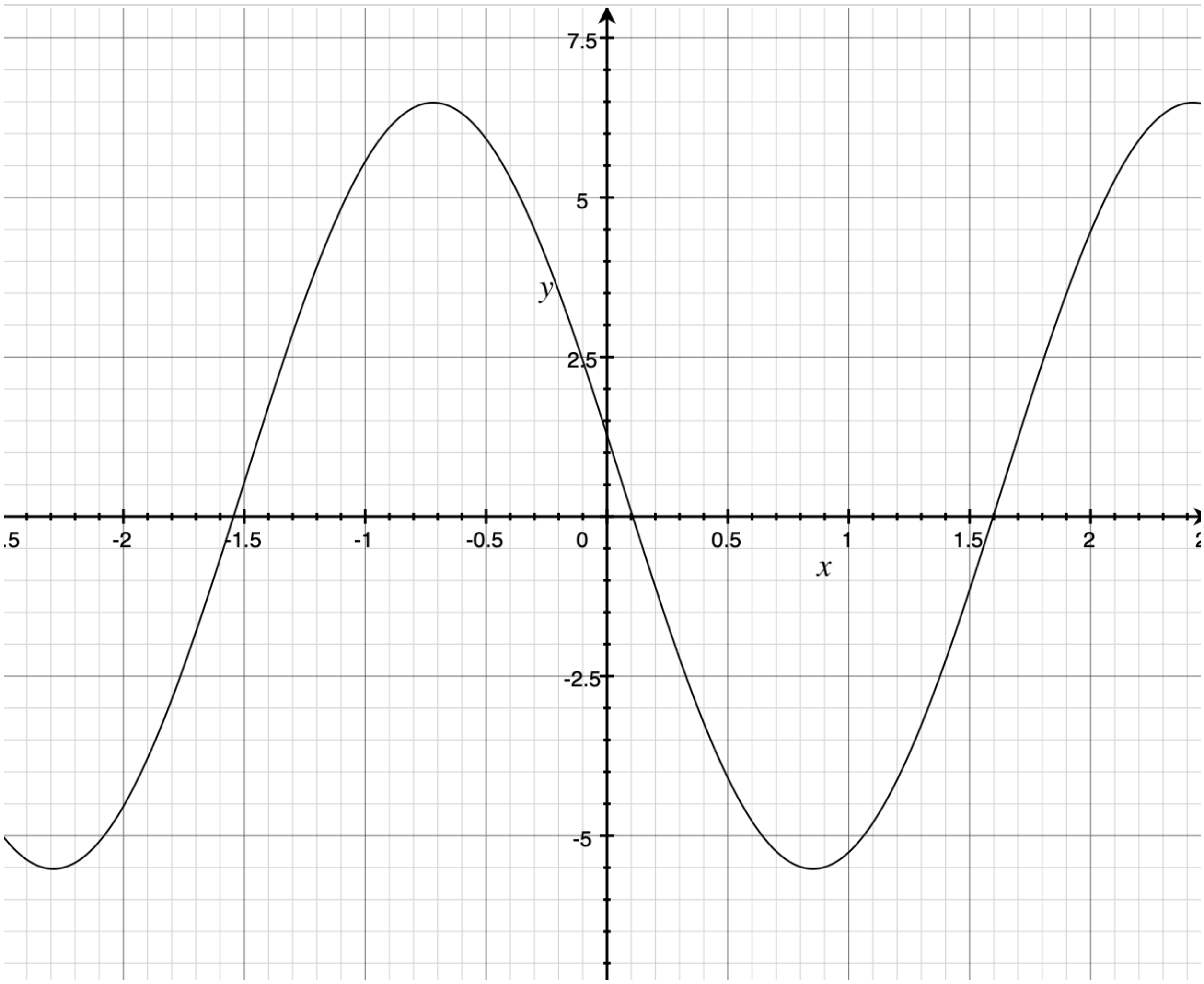
$$B = L = 11.9m$$

$$C = \frac{L^2 g}{2V^2} = (11.9m)^2 9.81\frac{m}{sec^2}\left[\frac{hr}{170km}\right]^2\frac{1}{2}\left[\frac{3600sec}{hr}\right]^2\left[\frac{km}{(10)^3 m}\right]^2$$

$$C = .3115m$$

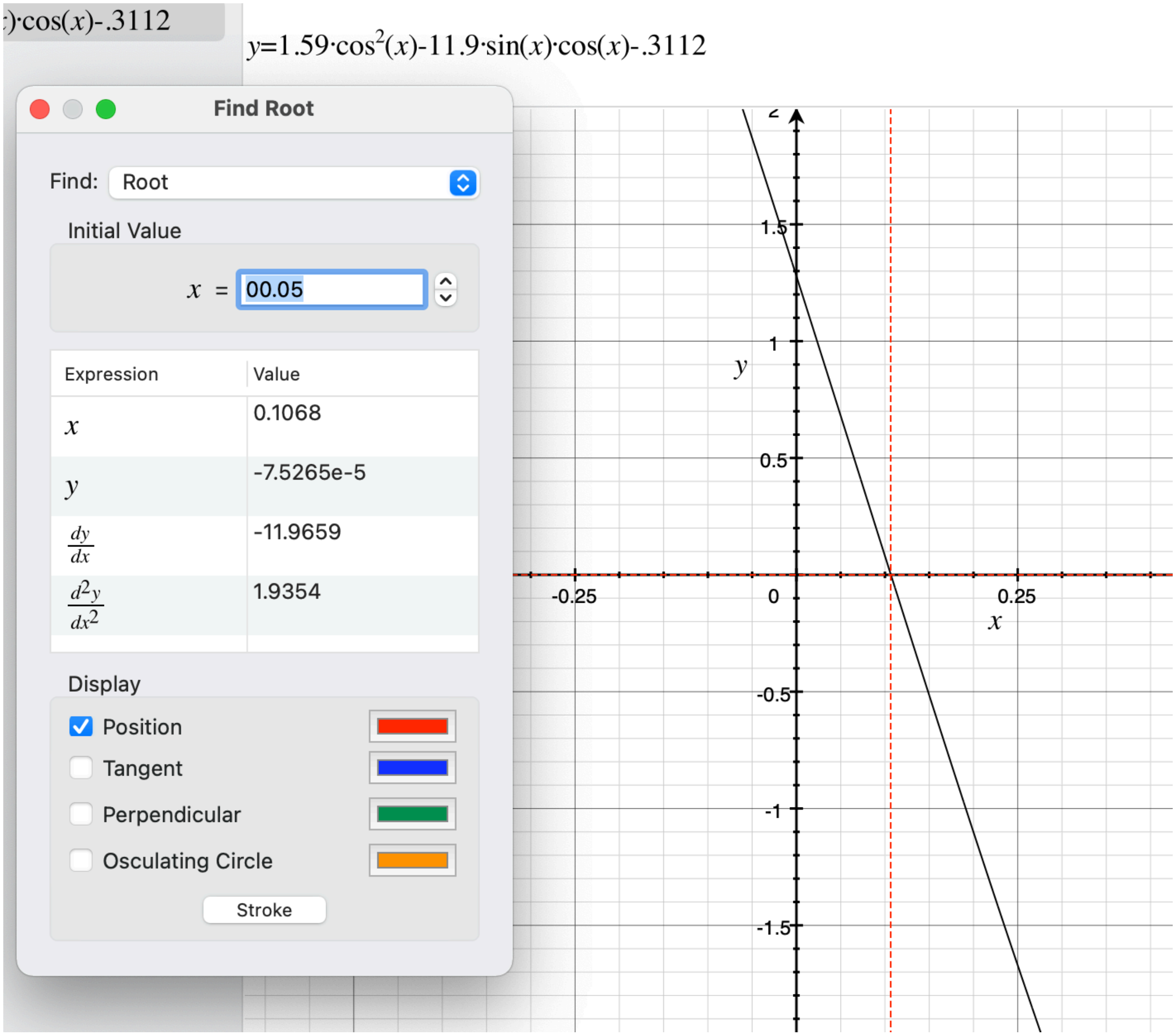
As another check, note that the units on all terms in (7) are meters and they cancel. Equation (7) is dimensionally correct.

$y=1.59 \cdot \cos^2(x) - 11.9 \cdot \sin(x) \cdot \cos(x) - .3112$



A graph of the function $f(\theta)$ is shown as a periodic function on the following page.

Here x in the equation is the angle θ. Note that the solution is periodic giving both positive and negative values for θ. The first negative value for θ corresponds to hitting the ball upward (not below the horizon). The ball will reach a peak and then graze the net on its way down. Not the solution we want. additional solutions correspond to measuring the angle θ $\pm 2n\pi$ radians. The plotting software I used (Grapher) can find the root of



the equation. In the next figure I zoom in and use the software to find the root.

The angle θ (represented by x in the software) is .1068 radians or 7.7 degrees (below the horizon). When the ball hits the court surface, y(t)=H.

(8) $$y(t) = V\sin(\theta)t + \frac{1}{2}gt^2$$

$$H = V\sin(.1068)t + \frac{1}{2}gt^2$$

$At^2 + Bt + C$ where A=$\frac{1}{2}g$, B=Vsin(.1068), and C=-H

$$A = \frac{1}{2}g = .5 * 9.8\frac{m}{sec^2} = 4.9\frac{m}{sec^2}$$

$$B = V\sin(.1068) = 170\frac{km}{hr}\frac{hr}{3600sec}\frac{1000m}{km}\sin(.1068) = 5.033\frac{m}{sec}$$

C=-H=-2.5m

Substituting A,B and C into the quadratic formula, we get

(9) $$t = \frac{-5.063\frac{m}{sec} \pm \sqrt{(-5.063\frac{m}{sec})^2 - 4 * 4.9\frac{m}{sec^2} * (-2.5m)}}{2 * 4.9\frac{m}{sec^2}}$$

t=.365 -1.40 sec

Obviously, we want the positive time. t=.365 sec. The next question is this in bounds? How far does the ball travel in the x direction in .365 sec. Returning to (1) and realizing that the horizontal velocity does not change. The horizontal velocity is given by

$$170\frac{km}{hr} * 1000\frac{m}{km}\frac{1}{3600}\frac{hr}{sec}cos(\theta) = 46.96\frac{m}{sec}$$

Thus the distance traveled is

$$46.96 * .365 = 17.14m$$

The end of the court from the point of service is 11.9+6.4=18.3 m. The ball, therefore, hits before the foul line.